

Operator inequalities among arithmetic mean, geometric mean and harmonic mean

Shigeru Furuichi*

Department of Information Science,
College of Humanities and Sciences, Nihon University,
3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan

Abstract. We give an upper bound for the weighted geometric mean using the weighted arithmetic mean and the weighted harmonic mean. We also give a lower bound for the weighted geometric mean. These inequalities are proven for two invertible positive operators.

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1 Introduction

Let \mathcal{H} be a complex Hilbert space. We represent the set of all bounded operators on \mathcal{H} by $B(\mathcal{H})$. If $A \in B(\mathcal{H})$ satisfies $A^* = A$, then A is called a self-adjoint operator. If a self-adjoint operator A satisfies $\langle x|A|x \rangle \geq 0$ for any $|x\rangle \in \mathcal{H}$, then A is called a positive operator. For two self-adjoint operators A and B , $A \geq B$ means $A - B \geq 0$. The notation $A > 0$ means A is an invertible positive operator.

It is well-known that we have the following Young inequalities for invertible positive operators A and B :

$$(1 - \nu)A + \nu B \geq A\#_{\nu}B \geq \{(1 - \nu)A^{-1} + \nu B^{-1}\}^{-1}, \quad (1)$$

where $A\#_{\nu}B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^{\nu}A^{1/2}$ represents the geometric mean for two positive operators A and B and a weighted parameter $\nu \in [0, 1]$ [1]. (In this paper, we use the notation $A\#B$ instead of $A\#_{1/2}B$ for the simplicity.) $(1 - \nu)A + \nu B$ and $\{(1 - \nu)A^{-1} + \nu B^{-1}\}^{-1}$ are called weighted arithmetic mean and harmonic mean for two positive operators, respectively. The simplified and elegant proof for the inequalities (1) was given in [2]. Recently, refinements of the inequalities (1) were given in our papers [3, 4]. It is also notable that improvements of [4] have been given in the paper [5]. And further improvements have been given in quite recent papers [6] and [7]. In this short note, we consider the relations among operator means for two positive operators.

We start from the following proposition.

Proposition 1.1 *Let A, B be invertible positive operators and r be a real number. Then we have the following inequalities.*

$$(i) \text{ If } r \geq 2, \text{ then } rA\#B + (1 - r)\frac{A+B}{2} \leq \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}.$$

*E-mail: furuichi@chs.nihon-u.ac.jp

(ii) If $r \leq 1$, then $rA\#B + (1-r)\frac{A+B}{2} \geq \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}$.

Proof: In general, by using the notion of the representing function $f_m(x) = 1mx$ for operator mean m , it is well-known [1] that $f_m(x) \leq f_n(x)$ holds for $x > 0$ if and only if $AmB \leq AnB$ holds for all positive operators A and B . Thus we can prove this proposition from the following scalar inequalities for $t > 0$.

$$(i) \quad r\sqrt{t} + (1-r)\frac{t+1}{2} \leq \frac{2t}{t+1}, \quad (r \geq 2).$$

$$(ii) \quad r\sqrt{t} + (1-r)\frac{t+1}{2} \geq \frac{2t}{t+1}, \quad (r \leq 1).$$

Actually (i) above can be proven in the following way. We set $f_r(t) \equiv \frac{2t}{t+1} - r\sqrt{t} - (1-r)\frac{t+1}{2}$, then $\frac{df_r(t)}{dt} = -\sqrt{t} + \frac{t+1}{2} \geq 0$ implies $f_r(t) \geq f_2(t)$ for $r \geq 2$. From the relation $\frac{2t}{t+1} + \frac{t+1}{2} \geq 2\sqrt{t}$, we have $f_2(t) \geq 0$. We also give the proof for (ii) above. We set $g_r(t) \equiv r\sqrt{t} + (1-r)\frac{t+1}{2} - \frac{2t}{t+1}$, then $\frac{dg_r(t)}{dt} = \sqrt{t} - \frac{t+1}{2} \leq 0$ implies $g_r(t) \geq g_1(t)$ for $r \leq 1$. From the relation $\frac{2t}{t+1} \leq \sqrt{t}$, we have $g_1(t) \geq 0$. ■

Remark 1.2 We have counter-examples of both inequalities (i) and (ii) in Proposition 1.1 for $1 < r < 2$. For example, we take $r = 1.5$. Then we have the following computations. $\frac{2t}{t+1} - r\sqrt{t} - (1-r)\frac{t+1}{2} \simeq 0.122302$ when $t = 0.01$ and $\frac{2t}{t+1} - r\sqrt{t} - (1-r)\frac{t+1}{2} \simeq -0.037987$ when $t = 2$.

2 Main results

Proposition 1.1 can be generalized by means of weighted parameter $\nu \in [0, 1]$, as the second inequality in (2) below.

Theorem 2.1 If (i) $0 \leq \nu \leq 1/2$ and $0 < A \leq B$ or (ii) $1/2 \leq \nu \leq 1$ and $0 < B \leq A$, then the following inequalities hold

$$A\#B + \left(\nu - \frac{1}{2}\right)(B - A) \leq A\#_\nu B \leq \frac{1}{2}\{(1-\nu)A + \nu B\} + \frac{1}{2}\{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1}. \quad (2)$$

Remark 2.2 Under the same conditions as in Theorem 2.1, we have $A\#B \geq A\#_\nu B$.

In order to prove Theorem 2.1, we firstly prove the corresponding scalar inequalities, as it was similarly done in Proposition 1.1.

Lemma 2.3 If (i) $0 \leq \nu \leq 1/2$ and $t \geq 1$ or (ii) $1/2 \leq \nu \leq 1$ and $0 < t \leq 1$, then the following inequalities hold

$$2\sqrt{t} + (2\nu - 1)(t - 1) \leq 2t^\nu \leq (1 - \nu) + \nu t + \left\{(1 - \nu) + \frac{\nu}{t}\right\}^{-1}. \quad (3)$$

Proof: It is trivial for the case $t = 1$. For the cases $\nu = 0, 1/2$ or 1 , the inequalities (3) hold. So we assume $t \neq 1$ and $\nu \neq 0, 1/2, 1$. We firstly prove the first inequality of the inequalities (3), under the condition (i) $0 < \nu < 1/2$ and $t > 1$ or (ii) $1/2 < \nu < 1$ and $0 < t < 1$. Here we put $f_\nu(t) \equiv t^\nu - \sqrt{t} - (\nu - \frac{1}{2})(t - 1)$. Then we have $f'_\nu(t) = \nu t^{\nu-1} - \frac{1}{2\sqrt{t}} - (\nu - \frac{1}{2})$ and $f'_\nu(1) = 0$. We also have $f''_\nu(t) = -\nu(1 - \nu)t^{\nu-2} + \frac{1}{4}t^{-3/2}$. Thus we have $f''_\nu(t) = 0 \Leftrightarrow t = t_\nu \equiv \{4\nu(1 - \nu)\}^{\frac{2}{1-2\nu}}$. We find $t_\nu < 1$ in the case $0 < \nu < 1/2$ and $t > 1$. Then we find

$f''_\nu(t) \geq 0$ for $t > 1 (> t_\nu)$. So $f'_\nu(t)$ is monotone increasing for $t > 1$ and we have $f'_\nu(1) = 0$. Thus we find $f'_\nu(t) \geq 0$ for $t > 1$. So $f_\nu(t)$ is monotone increasing for $t > 1$. Therefore we have $f_\nu(t) \geq f_\nu(1) = 0$. We also find $t_\nu > 1$ in the case $1/2 < \nu < 1$ and $0 < t < 1$. Then we find $f''_\nu(t) \geq 0$ for $0 < t < 1 (< t_\nu)$. So $f'_\nu(t)$ is monotone increasing for $0 < t < 1$ and we have $f'_\nu(1) = 0$. Thus we find $f'_\nu(t) \leq 0$ for $0 < t < 1$. So $f_\nu(t)$ is monotone decreasing for $0 < t < 1$. Therefore we have $f_\nu(t) \geq f_\nu(1) = 0$. Thus the proof for the first inequality of the inequalities (3) is done.

We prove the second inequality of the inequalities (3). We put $g_\nu(t) \equiv (1 - \nu) + \nu t + \frac{1}{1 - \nu + \frac{\nu}{t}} - 2t^\nu$. Then we have $g_\nu(t) = (1 - \nu) + \nu t + \frac{t}{(1 - \nu)t + \nu} - 2t^\nu \geq 2\sqrt{\frac{\{(1 - \nu) + \nu t\}t}{(1 - \nu)t + \nu}} - 2t^\nu$. Since $g_\nu(t) \geq 0$ is equivalent to $\frac{(1 - \nu) + \nu t}{(1 - \nu)t + \nu} \geq t^{2\nu - 1}$, we put again $h_\nu(t) \equiv (1 - \nu) + \nu t - \{(1 - \nu)t + \nu\}t^{2\nu - 1}$. Then we prove $h_\nu(t) > 0$ under the condition (i) $0 < \nu < 1/2$ and $t > 1$ or (ii) $1/2 < \nu < 1$ and $0 < t < 1$. By the elementary calculations, we have $h'_\nu(t) = \nu - 2\nu(1 - \nu)t^{2\nu - 1} - \nu(2\nu - 1)t^{2\nu - 2}$, $h'_\nu(1) = 0$ and $h''_\nu(t) = -2\nu(1 - \nu)(2\nu - 1)t^{2\nu - 3}(t - 1)$. Then we find $h''_\nu(t) = 0 \Leftrightarrow t = 1$. In the case $t > 1$, we have $h''_\nu(t) \geq 0$. So $h'_\nu(t)$ is monotone increasing for $t > 1$ and we have $h'_\nu(1) = 0$. Thus we have $h'_\nu(t) \geq 0$ for $t > 1$. So $h_\nu(t)$ is monotone increasing for $t > 1$. Thus we have $h_\nu(t) \geq h_\nu(1) = 0$. In the case $0 < t < 1$, we also have $h''_\nu(t) \geq 0$. So $h'_\nu(t)$ is monotone increasing for $0 < t < 1$ and we have $h'_\nu(1) = 0$. Thus we have $h'_\nu(t) \leq 0$ for $0 < t < 1$. So $h_\nu(t)$ is monotone decreasing for $0 < t < 1$. Thus we have $h_\nu(t) \geq h_\nu(1) = 0$. Thus the proof for the second inequality of the inequalities (3) is done. ■

Lemma 2.4 *Let $r \in \mathbb{R}$. Then the function $k_{r,\nu}(t) \equiv rt^\nu + (1 - r)\{(1 - \nu) + \nu t\}$, $(0 \leq \nu \leq 1, t > 0)$ is monotone decreasing with respect to r . Therefore, $k_{r,\nu}(t) \leq k_{2,\nu}(t)$ for $r \geq 2$ and $k_{r,\nu}(t) \geq k_{1,\nu}(t)$ for $r \leq 1$.*

Proof : The proof is done by $\frac{\partial k_{r,\nu}(t)}{\partial r} = t^\nu - \{(1 - \nu) + \nu t\} \leq 0$, for $\nu \in [0, 1]$ and $t > 0$. ■

Lemma 2.4 provides the following results.

Lemma 2.5 *Let $r \geq 2$. If (i) $0 \leq \nu \leq 1/2$ and $t \geq 1$ or (ii) $1/2 \leq \nu \leq 1$ and $0 < t \leq 1$, then*

$$rt^\nu + (1 - r)\{(1 - \nu) + \nu t\} \leq \left\{(1 - \nu) + \frac{\nu}{t}\right\}^{-1}.$$

Proof : The proof follows directly from Lemma 2.3 and Lemma 2.4. ■

Lemma 2.6 *Let $r \leq 1$. For $0 < \nu \leq 1$ and $t > 0$, we have*

$$rt^\nu + (1 - r)\{(1 - \nu) + \nu t\} \geq \left\{(1 - \nu) + \frac{\nu}{t}\right\}^{-1}.$$

Proof: For $r \leq 1$, it follows from Lemma 2.4 that $rt^\nu + (1 - r)\{(1 - \nu) + \nu t\} \geq t^\nu$. Since we have $t^\nu \geq \left\{(1 - \nu) + \frac{\nu}{t}\right\}^{-1}$, the proof is done. ■

Finally we have the following corollary.

Corollary 2.7 *Let $r \geq 2$. If (i) $0 < \nu \leq 1/2$ and $0 < A \leq B$ or (ii) $1/2 \leq \nu \leq 1$ and $0 < B \leq A$, then*

$$rA\#_\nu B + (1 - r)\{(1 - \nu)A + \nu B\} \leq \{(1 - \nu)A^{-1} + \nu B^{-1}\}^{-1}.$$

Let $r \leq 1$. For $0 < \nu < 1$ and $t > 0$, we have

$$rA\#_{\nu}B + (1-r)\{(1-\nu)A + \nu B\} \geq \{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1}.$$

Proof: The proof can be done applying Lemma 2.5, Lemma 2.6 and Theorem 2.1. ■

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